Toward the best possible Rosenthal-type bound

Iosif Pinelis*

Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan 49931, USA
E-mail: ipinelis@mtu.edu

Abstract: It is shown that, for any given $p \geqslant 5$, $A > 0$, and $B > 0$, the exact upper bound on $\mathsf{E}\,|\sum X_i|^p$ over all independent zero-mean random variables (r.v.'s) $X_1, \dots, X_n$ such that $\sum \mathsf{E}\,X_i^2 = B$ and $\sum \mathsf{E}\,|X_i|^p = A$ coincides with the exact upper bound on $\mathsf{E}\,|c_1\tilde\Pi_{\lambda_1} - c_2\tilde\Pi_{\lambda_2}|^p$ over all $c_1, c_2, \lambda_1, \lambda_2 > 0$ such that $c_1^p\lambda_1 + c_2^p\lambda_2 = A$ and $c_1^2\lambda_1 + c_2^2\lambda_2 = B$, where $\tilde\Pi_{\lambda_1}$ and $\tilde\Pi_{\lambda_2}$ are independent centered Poisson r.v.'s with parameters λ_1 and λ_2. A somewhat more general result is in fact obtained. As a tool used in the proof, a calculus of variations of generalized moments of infinitely divisible distributions with respect to variations of the Lévy characteristics is developed.

AMS 2010 subject classifications: Primary 60E15; secondary 60E07.
Keywords and phrases: Rosenthal inequality, bounds on moments, sums of independent random variables, probability inequalities, calculus of variations, infinitely divisible distributions, Lévy characteristics.

Contents

1. Introduction, summary, and discussion

Let $\mathscr{X}$ denote the set of all finite sequences $\mathbf{X} = (X_1, \dots, X_n)$ of independent zero-mean r.v.'s. For any $\mathbf{X} = (X_1, \dots, X_n) \in \mathscr{X}$, let

$$S_{\mathbf{X}} := X_1 + \dots + X_n. \tag{1}$$

*Supported by NSA grant H98230-12-1-0237

Take any real number $p \geqslant 2$ and any positive real numbers A and B. Consider

$$\begin{aligned}
\mathscr{X}_{p;A,B} &:= \Big\{\mathbf{X}=(X_1,\dots,X_n)\in\mathscr{X}\colon\ \sum_1^n \mathsf{E}\,X_i^2=B,\ \sum_1^n \mathsf{E}\,|X_i|^p=A\Big\},\\
\mathscr{X}_{p;\leqslant A,\leqslant B} &:= \Big\{\mathbf{X}=(X_1,\dots,X_n)\in\mathscr{X}\colon\ \sum_1^n \mathsf{E}\,X_i^2\leqslant B,\ \sum_1^n \mathsf{E}\,|X_i|^p\leqslant A\Big\},\\
\mathscr{E}_{p;A,B} &:= \sup\{\mathsf{E}\,|S_{\mathbf{X}}|^p\colon \mathbf{X}\in\mathscr{X}_{p;A,B}\},\\
\mathscr{E}_{p;\leqslant A,\leqslant B} &:= \sup\{\mathsf{E}\,|S_{\mathbf{X}}|^p\colon \mathbf{X}\in\mathscr{X}_{p;\leqslant A,\leqslant B}\}.
\end{aligned}$$

Rosenthal [17] showed that

$$\mathscr{E}_{p;A,B}\leqslant C_p\,(A+B),$$

with $C_p=2^{p^2}$. For subsequent developments, see e.g. [8, Sections 4 and 5], [9], [6], and references therein.

Using Jensen's inequality and homogeneity, it is easy to see that

$$\mathscr{E}_{p;\leqslant A,\leqslant B}=\mathscr{E}_{p;A,B}.$$

This in turn implies (cf. [6]) that the problem of finding $\mathscr{E}_p(A,B)$ is equivalent to that of finding, for an arbitrary balancing parameter $\gamma\in(0,\infty)$, the best constant $C_{p,\gamma}$ in the inequality

$$\mathsf{E}\,|S_{\mathbf{X}}|^p\leqslant C_{p,\gamma}\max(\gamma^pA,B)$$

for all $\mathbf{X}\in\mathscr{X}_{A,B}$. Indeed,

$$C_{p,\gamma}=\mathscr{E}_p(1/\gamma,1)\quad\text{and}\quad \mathscr{E}_p(A,B)=B^pC_{p,B/A}.$$

For any real $\lambda>0$, let Π_λ denote a r.v. with the Poisson distribution with parameter λ, and then introduce the corresponding centered r.v.

$$\tilde\Pi_\lambda:=\Pi_\lambda-\lambda.$$

Using Theorem 4 by Utev [18], Bestsennaya and Utev [1] showed that for $p=4,6,\dots$

$$\mathscr{E}_{p;A,B}=c^p\,\mathsf{E}\,|\tilde\Pi_\lambda|^p, \tag{2}$$

where

$$\lambda:=\lambda_p(A,B):=\Big(\frac{B}{A}\Big)^{\frac{2p}{p-2}}\quad\text{and}\quad c:=c_p(A,B):=\Big(\frac{A^p}{B^2}\Big)^{\frac{1}{p-2}}.$$

Obviously, if p is an even natural number, then the absolute pth moment $\mathsf{E}\,|X|^p$ of a r.v. X is the same as its pth moment $\mathsf{E}\,X^p$. This fact allows the proof in [1] to be based on the well-known representation of moments in terms of cumulants and the log-convexity of $\int_{\mathbb{R}}|x|^rG(\mathrm{d}x)$ in $r>0$, for any nonnegative measure G.

Under the additional restriction that the X_i's be symmetrically distributed, exact Rosenthal-type inequalities were obtained in [4, 5].

Take any r.v. X such that

$$\mathsf{E}\,X = 0 \quad \text{and} \quad \mathsf{E}\,e^{c|X|} < \infty \tag{3}$$

for some real $c > 0$ and consider

$$\mathscr{X}_{p;X;A,B} := \Big\{\mathbf{X} \in \mathscr{X}_{p;A,B} \colon \mathbf{X} \text{ is independent of } X\Big\},$$
$$\mathscr{X}_{p;X;\leqslant A,\leqslant B} := \Big\{\mathbf{X} \in \mathscr{X}_{p;\leqslant A,\leqslant B} \colon \mathbf{X} \text{ is independent of } X\Big\}.$$

The main result of the present paper is

Theorem 1.1. *Suppose that $p \geqslant 5$. Then*

$$\begin{aligned}
\sup_{\mathbf{X}\in\mathscr{X}_{p;X;\leqslant A,\leqslant B}} \mathsf{E}\,|X + S_{\mathbf{X}}|^p &= \sup_{\mathbf{X}\in\mathscr{X}_{p;X;A,B}} \mathsf{E}\,|X + S_{\mathbf{X}}|^p \\
&= \sup\{\mathsf{E}\,|X + c_1\tilde\Pi_{\lambda_1} - c_2\tilde\Pi_{\lambda_2}|^p \colon (c_1, c_2, \lambda_1, \lambda_2) \in (0,\infty)^4; \\
&\qquad c_1^2\lambda_1 + c_2^2\lambda_2 = B,\ c_1^p\lambda_1 + c_2^p\lambda_2 = A; \\
&\qquad X, \tilde\Pi_{\lambda_1}, \tilde\Pi_{\lambda_2} \textit{ are independent}\}.
\end{aligned} \tag{4}$$

Remark 1.2. It is of substantial interest to obtain exact Rosenthal-type inequalities for moment functions more general than the function $|\cdot|^p$ used in Theorem 1.1; cf. e.g. [3, 4]. In fact, one can indeed easily extend the result of Theorem 1.1 to the class of all moment functions of the form

$$x \longmapsto \int_{[5,p]\times[0,\infty)} (a+x)_+^r\,\nu_1(\mathrm{d}r\times\mathrm{d}a) + \int_{[5,p]\times[0,\infty)} (a-x)_+^r\,\nu_2(\mathrm{d}r\times\mathrm{d}a), \tag{5}$$

where ν_1 and ν_2 are any nonnegative Borel measures on the set $[5,p]\times[0,\infty)$ such that the resulting moment function is real-valued; of course, the moment function $x \mapsto |x|^p \big(= x_+^p + (-x)_+^p\big)$ is just one member of this class; as usual, we let $x_+ := 0 \vee x$ and $x_+^r := (x_+)^r$ for all real x and all real $r > 0$. To see why this extension of Theorem 1.1 is valid, one needs to look at the place in the proof of the theorem that imposes the narrowest restriction on the moment function – and that is the inequality $g^{(4)}(su) - u^{p-4}g^{(4)}(s) > 0$ for $u \in (0,1)$ in (74). The class of functions given by (5) may be compared with classes of moment functions considered e.g. in [7, 10, 12].

2. Proof of Theorem 1.1

2.1. Domination by the accompanying compound Poisson distribution

Theorem A. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be any twice continuously differentiable function such that f and f'' are convex. Let G is any finite nonnegative Borel measure*

on $\mathbb{R}$ such that $G(\{0\}) = 0$ and $\int_{\mathbb{R}} xG(\mathrm{d}x) = 0$, and then let X_G be any r.v. with the characteristic function $t \mapsto \exp \int_{\mathbb{R}}(e^{itx} - 1)G(\mathrm{d}x)$. Then

$$\sup\{\mathsf{E}\, f(S_{\mathbf{X}}) \colon \mathbf{X} \in \mathscr{X}, G_{\mathbf{X}} = G\} = \mathsf{E}\, f(X_G),$$

where $S_{\mathbf{X}}$ is as in (1) and $G_{\mathbf{X}}$ is the "sum of the tails" measure defined by

$$G_{\mathbf{X}}(E) := \sum \mathsf{P}(X_i \in E \setminus \{0\})$$

for all Borel subsets E of $\mathbb{R}$. In particular, for all $x \in \mathbb{R}$ and all real $p \geqslant 3$

$$\sup\{\mathsf{E}\,|S_{\mathbf{X}} - x|^p \colon \mathbf{X} \in \mathscr{X}, G_{\mathbf{X}} = G\} = \mathsf{E}\,|X_G - x|^p,$$
$$\sup\{\mathsf{E}(S_{\mathbf{X}} - x)_+^p \colon \mathbf{X} \in \mathscr{X}, G_{\mathbf{X}} = G\} = \mathsf{E}(X_G - x)_+^p.$$

Theorem A is essentially the same as the mentioned Theorem 4 by Utev [18] (cf. [11,14,16]). The assumptions on f in [18, Theorem 4] were slightly different; namely, it was assumed there that f'' is convex whereas f is nonnegative and satisfies a certain limited growth condition, which latter may be dropped, by [12, Proposition 1 and Lemma 4], provided that f and f'' are convex, as in Theorem A.

Remark. If a r.v. X has a finite expectation and a function $f\colon \mathbb{R} \to \mathbb{R}$ is convex, then, by Jensen's inequality, $\mathsf{E}\, f(X)$ always exists in $(-\infty, \infty]$.

2.2. *Zero-mean truncation of zero-mean r.v.'s*

Proposition 2.1. *Let Y be any zero-mean r.v. Then for any real $M > 0$ there is a r.v. Y_M with the following properties:*

(i) $\mathsf{E}\, Y_M = 0$*;*
(ii) $|Y_M| \leqslant M \wedge |Y|$*;*
(iii) $Y_M \to Y$ *almost surely (a.s.) as* $M \to \infty$*.*

This follows immediately from [13, Proposition 3.15] on letting

$$Y_M := Y\, \mathbf{I}\{|Y| \leqslant M, |\mathsf{r}(Y, U)| \leqslant M\},$$

where U any r.v. which is independent of Y and uniformly distributed on the unit interval $[0, 1]$, whereas r stands for the reciprocating function of (the distribution of) the r.v. Y, in accordance with the definition [13, (2.6)]; note that, by [13, Proposition 3.6], $|\mathsf{r}(Y, U)| < \infty$ a.s. The r.v. U, which may be referred to as a randomizing r.v., is used to split atoms of the distribution of Y, as such splitting may be needed to satisfy the condition $\mathsf{E}\, Y_M = 0$.

2.3. *A calculus of variations of generalized moments of infinitely divisible distributions with respect to variations of the Lévy characteristics*

Take any finite nonnegative Borel measure G on $\mathbb{R}$ with a finite value of

$$m_G := \int_{\mathbb{R}} uG(\mathrm{d}u);$$

at that neither one of the conditions $G(\{0\}) = 0$ or $\int_{\mathbb{R}} xG(\mathrm{d}x) = 0$, used in Theorem A, is required. Introduce the corresponding Borel probability measure μ_G on $\mathbb{R}$ by the condition

$$\int_{\mathbb{R}} h \mathrm{d}\mu_G = e^{-G(\mathbb{R})} \sum_{j=0}^{\infty} \frac{1}{j!} \int_{\mathbb{R}} h(u - m_G) G^{*j}(\mathrm{d}u) \tag{6}$$

for all functions $h \colon \mathbb{R} \to \mathbb{C}$ that are nonnegative or bounded, where G^{*j} is the j-fold convolution of G, with G^{*0} defined of course as the Dirac measure at 0. Then μ_G is the probability distribution of any r.v. X_G such that

$$\mathsf{E}\, e^{itX_G} = \exp \int_{\mathbb{R}} (e^{itx} - 1 - itx) G(\mathrm{d}x) \tag{7}$$

for all $t \in \mathbb{R}$; to check this, substitute the function $x \mapsto e^{itx}$ for h in (6). Clearly, this definition of X_G is in agreement with that in Theorem A, where it was additionally assumed that $G(\{0\}) = 0$ and $\int_{\mathbb{R}} xG(\mathrm{d}x) = 0$.

Lemma 2.2. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be any continuously differentiable function such that*

$$|f(x)| + |f'(x)| \leqslant Ce^{c|x|} \tag{8}$$

for some positive real numbers C and c and all $x \in \mathbb{R}$. Let G be a nonnegative Borel measure on $\mathbb{R}$. Let t_0 be a positive real number and let Δ be any finite signed Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} |u\Delta(\mathrm{d}u)| < \infty$, the measure

$$G_t := G + t\Delta \tag{9}$$

is nonnegative for all $t \in [0, t_0]$, and

$$\int_{\mathbb{R}} e^{c|x|} G_t(\mathrm{d}x) \leqslant U < \infty \tag{10}$$

for some real $c > 0$ and $U > 0$ and all $t \in [0, t_0]$ (or, equivalently, for $t \in \{0, t_0\}$). Let

$$g_t(x) := g_{f;t}(x) := \mathsf{E}\, f(x + X_{G_t}) = \int_{\mathbb{R}} f(u + x) \mu_{G_t}(\mathrm{d}u) \tag{11}$$

for $t \in [0, t_0]$ and $x \in \mathbb{R}$, with

$$g := g_0.$$

Then $g_t(x)$ is well defined and finite for all $t \in [0, t_0]$ and $x \in \mathbb{R}$, and

$$\frac{\mathrm{d}g_t(0)}{\mathrm{d}t}\Big|_{t=0+} := \lim_{t \downarrow 0} \frac{g_t(0) - g(0)}{t} = \int_{\mathbb{R}} [g(u) - g(0) - g'(0)u] \Delta(\mathrm{d}u). \tag{12}$$

Lemma 2.3. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be any twice continuously differentiable function such that*

$$|f(x)| + |f'(x)| + |f''(x)| \leqslant Ce^{c|x|} \tag{13}$$

for some positive real numbers C and c and all $x \in \mathbb{R}$. Then

$$\frac{\mathrm{d}\,\mathsf{E}\,f(\sigma Z)}{\mathrm{d}\sigma} = \sigma\,\mathsf{E}\,f''(\sigma Z) \quad \text{or, equivalently,} \quad \frac{\mathrm{d}\,\mathsf{E}\,f(\sigma Z)}{\mathrm{d}(\sigma^2)} = \tfrac{1}{2}\,\mathsf{E}\,f''(\sigma Z) \tag{14}$$

for all real $\sigma > 0$; here and subsequently,

$$Z \sim N(0,1).$$

The second equality in (14) holds as well for $\sigma = 0$ in the sense that

$$\lim_{\sigma\downarrow 0} \frac{\mathsf{E}\,f(\sigma Z) - f(0)}{\sigma^2} = \tfrac{1}{2}\,\mathsf{E}\,f''(0).$$

Of course, one can recognize the heat equation in (14).

Lemmas 2.2 and 2.3 can be combined into

Lemma 2.4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be any twice continuously differentiable function such that (13) holds for some positive real numbers C and c and all $x \in \mathbb{R}$. Let G, t_0, Δ, g_t, and g be as in Lemma 2.2. Take any $\sigma \in [0,\infty)$ and $\kappa \in (-\sigma^2/t_0, \infty)$. Then*

$$\begin{aligned} h_t(x) := h_{f;t}(x) := \mathsf{E}\,g_{f;t}\big(x + Z\sqrt{\sigma^2+\kappa t}\,\big) &= \mathsf{E}\,f\big(x + X_{G_t} + Z\sqrt{\sigma^2+\kappa t}\,\big) \\ &= \int_{\mathbb{R}} \mathsf{E}\,f\big(u + x + Z\sqrt{\sigma^2+\kappa t}\,\big)\mu_{G_t}(\mathrm{d}u) \end{aligned} \tag{15}$$

is well defined and finite for all $t \in [0,t_0]$ and $x \in \mathbb{R}$, and

$$\frac{\mathrm{d}h_t(0)}{\mathrm{d}t}\Big|_{t=0+} := \lim_{t\downarrow 0} \frac{h_t(0) - h(0)}{t} = \tfrac{\kappa}{2} h''(0) + \int_{\mathbb{R}} [h(u) - h(0) - h'(0)u]\Delta(\mathrm{d}u), \tag{16}$$

where

$$h := h_0.$$

Lemma 2.4 is reminiscent of the well-known result on the infinitesimal generator of a Lévy process but is in a sense more general:

(i) in contrast with the Lévy process case, no semigroup property can be exploited in Lemma 2.4; more specifically, the integral in (16) is with respect to the directional derivative Δ of the Lévy measure $G_t = G + t\Delta$ rather than with respect to the Lévy measure G itself – indeed, the Lévy process case corresponds to the special choice of $\Delta = G$;

(ii) in contrast with the growth-and-smoothness condition (13), assumed in Lemma 2.4, the moment functions in the domain of the infinitesimal generator of a Lévy process are usually required to be infinitely smooth and vanishing at $\pm\infty$ together with all the derivatives.

Proof of Lemma 2.2. To begin, note that m_{G_t} is well defined and finite, by (10); in fact,

$$\max_{t\in[0,t_0]} |m_{G_t}| = \max(|m_{G_0}|, |m_{G_{t_0}}|) =: m_* < \infty. \tag{17}$$

Without loss of generality (w.l.o.g.), $t_0 = 1$ – since Δ and t in the definition (9) of G_t can be replaced by $t_0\Delta$ and t/t_0, respectively. Next, by (8) and (10),

$$\int_{\mathbb{R}} f(u + x - m_{G_t}) G_t^{*j}(\mathrm{d}u) \leqslant C e^{c(|x|+m_*)} \int_{\mathbb{R}} e^{c|u|} G_t^{*j}(\mathrm{d}u) \leqslant C e^{c(|x|+m_*)} U^j \tag{18}$$

for all $j \in \{0, 1, \dots\}$; here and elsewhere in this proof, it is assumed that t and x are arbitrary points in $[0, 1]$ and $\mathbb{R}$, respectively, unless otherwise stated. So, by (11) and (6),

$$|g_t(x)| \leqslant C e^{c(|x|+m_*)+U} < \infty. \tag{19}$$

Introduce

$$\tilde{g}_t(x) := e^{G_t(\mathbb{R})} g_t(x) \quad \text{and} \quad \tilde{g}(x) := \tilde{g}_0(x) = e^{G(\mathbb{R})} g(x). \tag{20}$$

Then, again by (11) and (6),

$$\tilde{g}_t(0) - \tilde{g}(0) = R_1(t) + R_2(t) + R_3(t), \tag{21}$$

where

$$\begin{aligned}
R_1(t) &:= \sum_{j=0}^{\infty} \frac{1}{j!} \int_{\mathbb{R}} [f(u - m_{G_t}) - f(u - m_G)] G^{*j}(\mathrm{d}u), \\
R_2(t) &:= \sum_{j=0}^{\infty} \frac{1}{j!} \int_{\mathbb{R}} f(u - m_G)(G_t^{*j} - G^{*j})(\mathrm{d}u), \\
R_3(t) &:= \sum_{j=0}^{\infty} \frac{1}{j!} \int_{\mathbb{R}} [f(u - m_{G_t}) - f(u - m_G)](G_t^{*j} - G^{*j})(\mathrm{d}u).
\end{aligned}$$

Next,

$$m_{G_t} = m_G + t \int_{\mathbb{R}} v\Delta(\mathrm{d}v)$$

and hence for all $u \in \mathbb{R}$

$$\frac{f(u - m_{G_t}) - f(u - m_G)}{t} \underset{t\downarrow 0}{\longrightarrow} -f'(u - m_G) \int_{\mathbb{R}} v\Delta(\mathrm{d}v);$$

moreover,

$$\begin{aligned}
\frac{|f(u - m_{G_t}) - f(u - m_G)|}{t} &= \Big| \int_0^1 \big[f'\big(u - m_G - st \textstyle\int_{\mathbb{R}} v\Delta(\mathrm{d}v)\big)\big] \mathrm{d}s \displaystyle\int_{\mathbb{R}} v\Delta(\mathrm{d}v) \Big| \\
&\leqslant \int_0^1 \big|f'\big(u - m_G - st \textstyle\int_{\mathbb{R}} v\Delta(\mathrm{d}v)\big)\big| \mathrm{d}s \displaystyle\int_{\mathbb{R}} |v\Delta(\mathrm{d}v)| \\
&\leqslant C e^{c(|u|+3m_*)} \times 2m_*
\end{aligned} \tag{22}$$

for all $t \in (0,1]$, the latter inequality following by (8) and (17), since $|\int_{\mathbb{R}} v\Delta(\mathrm{d}v)| = |m_{G_1} - m_G| \leqslant 2m_*$. So, by dominated convergence $\big($cf. (18) and (19)$\big)$ and in view of (6) and (11),

$$\begin{aligned}\lim_{t\downarrow 0}\frac{R_1(t)}{t} &= -\sum_{j=0}^{\infty}\frac{1}{j!}\int_{\mathbb{R}} f'(u-m_G)G^{*j}(\mathrm{d}u)\int_{\mathbb{R}} v\Delta(\mathrm{d}v) \\ &= -e^{G(\mathbb{R})}\int_{\mathbb{R}} f'\mathrm{d}\mu_G\int_{\mathbb{R}} v\Delta(\mathrm{d}v) = e^{G(\mathbb{R})}\int_{\mathbb{R}}[-g'(0)u]\Delta(\mathrm{d}u).\end{aligned} \tag{23}$$

Further,

$$R_2 = R_{21} + R_{22}, \tag{24}$$

where

$$\begin{aligned}R_{21}(t) &:= \sum_{j=1}^{\infty}\frac{1}{j!}\int_{\mathbb{R}} f(u-m_G)\, jt\,(G^{*(j-1)}*\Delta)(\mathrm{d}u) \\ &= t\sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{R}} f(u-m_G)\,(G^{*k}*\Delta)(\mathrm{d}u) \\ &= t\sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u+v-m_G)\,G^{*k}(\mathrm{d}u)\,\Delta(\mathrm{d}v) \\ &= t\,e^{G(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u+v)\mu_G(\mathrm{d}u)\,\Delta(\mathrm{d}v) \\ &= t\,e^{G(\mathbb{R})}\int_{\mathbb{R}} g(v)\Delta(\mathrm{d}v),\end{aligned} \tag{25}$$

$$R_{22}(t) := \sum_{j=2}^{\infty}\frac{1}{j!}\int_{\mathbb{R}} f(u-m_G)\sum_{i=2}^{j}\binom{j}{i}t^i\,(G^{*(j-i)}*\Delta^{*i})(\mathrm{d}u),$$

so that

$$\begin{aligned}\frac{|R_{22}(t)|}{t^2} &\leqslant C\sum_{j=0}^{\infty}\frac{1}{j!}\int_{\mathbb{R}} e^{c(|u|+|m_*|)}\sum_{i=0}^{j}\binom{j}{i}(G^{*(j-i)}*|\Delta|^{*i})(\mathrm{d}u) \\ &\leqslant Ce^{c|m_*|}\sum_{j=0}^{\infty}\frac{1}{j!}\Big(\int_{\mathbb{R}} e^{c|u|}(G+|\Delta|)(\mathrm{d}u)\Big)^j \\ &\leqslant Ce^{cm_*}\,\exp\Big\{\int_{\mathbb{R}} e^{c|u|}(G+|\Delta|)(\mathrm{d}u)\Big\} \leqslant Ce^{cm_*+3U} < \infty\end{aligned} \tag{26}$$

for all $t \in (0,1]$, by (10).

The term R_3 is bounded similarly. Here we recall (22) and write

$$
\begin{aligned}
&\frac{|R_3(t)|}{t^2} \\
&\leqslant \sum_{j=1}^{\infty}\frac{1}{j!}\int_{\mathbb{R}}\frac{|f(u-m_{G_t})-f(u-m_G)|}{t}\sum_{i=1}^{j}\binom{j}{i}t^{i-1}(G^{*(j-i)}*|\Delta|^{*i})(\mathrm{d}u) \\
&\leqslant 2Cm_*e^{3cm_*}\sum_{j=0}^{\infty}\frac{1}{j!}\int_{\mathbb{R}}e^{c|u|}\sum_{i=1}^{j}\binom{j}{i}(G^{*(j-i)}*|\Delta|^{*i})(\mathrm{d}u) \\
&\leqslant 2Cm_*e^{3cm_*}\exp\Big\{\int_{\mathbb{R}}e^{c|u|}(G+|\Delta|)(\mathrm{d}u)\Big\}\leqslant 2Cm_*e^{3cm_*+3U}<\infty
\end{aligned}
\tag{27}
$$

for all $t\in(0,1]$.

Collecting now (21), (23), (24), (25), (26), and (27), one finds

$$
\frac{\mathrm{d}\tilde g_t(0)}{\mathrm{d}t}\Big|_{t=0+}:=\lim_{t\downarrow 0}\frac{\tilde g_t(0)-\tilde g(0)}{t}=e^{G(\mathbb{R})}\int_{\mathbb{R}}[g(u)-g'(0)u]\Delta(\mathrm{d}u). \tag{28}
$$

On the other hand, recalling the definition (20) of $\tilde g_t$ and writing $G_t(\mathbb{R})=G(\mathbb{R})+\int_{\mathbb{R}}\Delta(\mathrm{d}u)$, one has

$$
\frac{\mathrm{d}g_t(0)}{\mathrm{d}t}\Big|_{t=0+}=\frac{\mathrm{d}\tilde g_t(0)}{\mathrm{d}t}\Big|_{t=0+}e^{-G(\mathbb{R})}-g(0)\int_{\mathbb{R}}\Delta(\mathrm{d}u).
$$

Therefore, (12) follows from (28). ☐

Proof of Lemma 2.3. Note that $\frac{\partial}{\partial\sigma}\big(\frac{1}{\sigma}\,\varphi(\frac{x}{\sigma})\big)=(\frac{\partial}{\partial x})^2\big(\varphi(\frac{x}{\sigma})\big)$ for all $(\sigma,x)\in(0,\infty)\times\mathbb{R}$, where φ is the density function of Z; so, in view of (13),

$$
\begin{aligned}
\frac{\mathrm{d}\,\mathsf{E}\,f(\sigma Z)}{\mathrm{d}\sigma}&=\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_{\mathbb{R}}f(x)\,\frac{1}{\sigma}\,\varphi\Big(\frac{x}{\sigma}\Big)\,\mathrm{d}x=\int_{\mathbb{R}}f(x)\,\frac{\partial}{\partial\sigma}\Big(\frac{1}{\sigma}\,\varphi\Big(\frac{x}{\sigma}\Big)\Big)\,\mathrm{d}x \\
&=\int_{\mathbb{R}}f(x)\,\Big(\frac{\partial}{\partial x}\Big)^2\Big(\varphi\Big(\frac{x}{\sigma}\Big)\Big)\,\mathrm{d}x=\int_{\mathbb{R}}f''(x)\,\varphi\Big(\frac{x}{\sigma}\Big)\,\mathrm{d}x=\sigma\,\mathsf{E}\,f''(\sigma Z),
\end{aligned}
$$

where we twice integrated by parts. The statement for $\sigma=0$ follows by the mean value theorem and dominated convergence. ☐

Proof of Lemma 2.4. In this proof, we take $(t,s)\in(0,t_0)^2:=(0,t_0)\times(0,t_0)$ and $x\in\mathbb{R}$, unless specified otherwise, and, as in the proof of Lemma 2.2, assume w.l.o.g. that $t_0=1$. Introduce

$$
\tilde h_{t,s}(x):=\mathsf{E}\,g_{f;t}\big(x+Z\sqrt{\sigma^2+\kappa s}\,\big)=g_{f_s;t}(x), \tag{29}
$$

where $g_{f;t}$ is as in (11) and

$$
f_s(x):=\mathsf{E}\,f\big(x+Z\sqrt{\sigma^2+\kappa s}\,\big). \tag{30}
$$

In view of (13) and dominated convergence,

$$|f_s(x)| + |f_s'(x)| + |f_s''(x)| \leqslant C_1 e^{c|x|}, \tag{31}$$

where $C_1 := C\,\mathsf{E}\, e^{c\sqrt{\sigma^2+\kappa}\,|Z|} < \infty$. So, by Lemma 2.2 with f_s in place of f,

$$\frac{\partial}{\partial t}\tilde{h}_{t,s}(0) = \int_{\mathbb{R}} [\tilde{h}_{t,s}(u) - \tilde{h}_{t,s}(0) - \tilde{h}_{t,s}'(0)u]\Delta(\mathrm{d}u). \tag{32}$$

It also follows from (31) that

$$|\tilde{h}_{t,s}(x)| + |\tilde{h}_{t,s}'(x)| \leqslant C_1 e^{c(|x|+m_*)+U} < \infty \tag{33}$$

(cf. (19)). On the other hand, by (10), $\int_{\mathbb{R}} e^{c|x|}\Delta(\mathrm{d}x) < \infty$. Now (32) implies that $\tilde{h}_{t,s}(0)$ is Lipschitz in t, for each s. On the other hand, $\tilde{h}_{t,s}(0)$ is continuous in s, for each t (which follows by dominated convergence from (29) and the continuity of $f_s(x)$ in s, which latter in turn follows by dominated convergence from (30)). Being Lipschitz in t and continuous in s, the expression $\tilde{h}_{t,s}(0)$ is continuous in (t,s) (cf. Minilemma 2.5 below); similarly, this conclusion holds for $\tilde{h}_{t,s}(u)$ (for each $u \in \mathbb{R}$) and also for $\tilde{h}_{t,s}'(0)$. Now the same conclusion follows for $\frac{\partial}{\partial t}\tilde{h}_{t,s}(0)$, by (32) and dominated convergence.

It also follows by (11), (13), and dominated convergence that

$$g_{f;t}^{(j)}(x) = \int_{\mathbb{R}} f^{(j)}(u+x)\mu_{G_t}(\mathrm{d}u)$$

for $j \in \{0,1,2\}$, where ${}^{(j)}$ denotes the jth derivative. This in turn implies

$$|g_{f;t}(x)| + |g_{f;t}'(x)| + |g_{f;t}''(x)| \leqslant C e^{c(|x|+m_*)+U}$$

(cf. (19)). So, by (29), Lemma 2.3, and dominated convergence,

$$\frac{\partial}{\partial s}\tilde{h}_{t,s}(0) = \tfrac{\kappa}{2}\,\mathsf{E}\, g_{f;t}''\big(Z\sqrt{\sigma^2+\kappa s}\,\big) = \tfrac{\kappa}{2}\,\tilde{h}_{t,s}''(0). \tag{34}$$

So, by the continuity of $\frac{\partial}{\partial t}\tilde{h}_{t,s}(0)$ in (t,s) and Minilemma 2.5 below, $\tilde{h}_{t,s}(0)$ is right-differentiable in (t,s) at $(0,0)$. Now Lemma 2.4 follows by (32), (34), and the relations $h_t = \tilde{h}_{t,t}$ and $h = h_0 = \tilde{h}_{0,0}$. $\square$

Minilemma 2.5. *Suppose that a function $f\colon [0,1)^2 \to \mathbb{R}$ is continuous, the right partial derivative $f_x'(0+,0) := \lim_{x\downarrow 0}\frac{f(x,0)-f(0,0)}{x}$ exists, the partial derivative $f_y'(x,y) := \frac{\partial}{\partial y}f(x,y)$ exists for $(x,y)\in(0,1)^2$, and the limit $f_y'(0+,0+) := \lim_{x,y\downarrow 0} f_y'(x,y)$ exists. Then f is right-differentiable at $(0,0)$ in the sense that*

$$f(x,y) - f(0,0) = f_x'(0+,0)x + f_y'(0+,0+)y + o(x+y)$$

as $x, y \downarrow 0$.

Proof of Minilemma 2.5. By the mean value theorem, for each $(x, y) \in (0,1)^2$ there exists some $\theta = \theta_{x,y} \in (0,1)$ such that $f(x,y) - f(x,0) = f'_y(x,\theta y)y$; hence,

$$\begin{aligned} f(x,y) - f(0,0) &= f(x,0) - f(0,0) + f(x,y) - f(x,0) \\ &= f'_x(0+,0)x + o(x) + f'_y(x,\theta y)y \\ &= f'_x(0+,0)x + f'_y(0+,0+)y + o(x+y) \end{aligned}$$

as $x, y \downarrow 0$. □

2.4. Main propositions in the proof of Theorem 1.1

Let $\mathscr{H}$ denote the set of all nonnegative Borel measures on $\mathbb{R}$. Take any real numbers $p > 3$, $A > 0$, $B > 0$, and $M > 0$ and introduce the following subsets of the set $\mathscr{H}$:

$$\mathscr{H}_{p,A,B} := \{H \in \mathscr{H} \colon \textstyle\int H(\mathrm{d}x) = B, \int |x|^{p-2} H(\mathrm{d}x) = A\}, \tag{35}$$
$$\mathscr{H}_{p,\leqslant A,\leqslant B} := \{H \in \mathscr{H} \colon \textstyle\int H(\mathrm{d}x) \leqslant B, \int |x|^{p-2} H(\mathrm{d}x) \leqslant A\}, \tag{36}$$
$$\mathscr{H}_{p,A,B;M} := \{H \in \mathscr{H}_{p,A,B} \colon \operatorname{supp} H \subseteq [-M, M]\}, \tag{37}$$
$$\mathscr{H}_{p,\leqslant A,\leqslant B;M} := \{H \in \mathscr{H}_{p,\leqslant A,\leqslant B} \colon \operatorname{supp} H \subseteq [-M, M]\}, \tag{38}$$

where $\operatorname{supp} H$ stands for the support set of the measure H; we also write $\int$ for $\int_{\mathbb{R}}$. Note that the set $\mathscr{H}_{p,\leqslant A,\leqslant B}$ obviously contains the other three of the above four sets.

Remark 2.6. Given any positive real A, B, and M, for the condition $\mathscr{H}_{p,A,B;M} \neq \emptyset$ to hold it is clearly necessary that

$$A \leqslant BM^{p-2} \tag{39}$$

or, equivalently, $B \geqslant A/M^{p-2}$. Therefore, in the statements concerning $\mathscr{H}_{p,A,B;M}$, let us assume by default that this restriction on A, B, and M holds.

Next, for any convex function $f \colon \mathbb{R} \to \mathbb{R}$ let

$$\mathscr{S}_{p,A,B}(f) := \sup\{\mathsf{E}\, f(X + Y_H) \colon H \in \mathscr{H}_{p,A,B}\}, \tag{40}$$
$$\mathscr{S}_{p,\leqslant A,\leqslant B}(f) := \sup\{\mathsf{E}\, f(X + Y_H) \colon H \in \mathscr{H}_{p,\leqslant A,\leqslant B}\}, \tag{41}$$
$$\mathscr{S}_{p,A,B;M}(f) := \sup\{\mathsf{E}\, f(X + Y_H) \colon H \in \mathscr{H}_{p,A,B;M}\}, \tag{42}$$
$$\mathscr{S}_{p,\leqslant A,\leqslant B;M}(f) := \sup\{\mathsf{E}\, f(X + Y_H) \colon H \in \mathscr{H}_{p,\leqslant A,\leqslant B;M}\}, \tag{43}$$

where X is a r.v. as in (3), Y_H is any r.v. which is independent of X and such that

$$\mathsf{E}\, e^{itY_H} = \eta_H(t) := \exp\big\{ -t^2 \textstyle\int e_2(itx)\, H(\mathrm{d}x)\big\} \tag{44}$$

for all real t, and

$$e_2(w) := \begin{cases} \frac{1}{w^2}(e^w - 1 - w) & \text{if } w \neq 0, \\ \frac{1}{2} & \text{if } w = 0. \end{cases} \tag{45}$$

Note that $\mathsf{E}\, Y_H^2 = -\eta_H''(0) = \int H(\mathrm{d}x) \leqslant B$ for all $H \in \mathscr{H}_{p,\leqslant A,\leqslant B}$ and hence for all H in any one of the sets defined in (35)–(38). Therefore, $\mathsf{E}\,|Y_H| < \infty$ and hence $\mathsf{E}\, f(X + Y_H)$ exists in $(-\infty, \infty]$ for all H in any one of the sets in (35)–(38) and all convex functions $f\colon \mathbb{R} \to \mathbb{R}$. Thus, each of the four suprema defined in (40)–(43) is correctly defined. Moreover, one has

Proposition 2.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be any convex function such that*

$$|f(x)| \leqslant Ce^{c|x|} \tag{46}$$

for some positive real numbers C and c and all $x \in \mathbb{R}$. Then the supremum $\mathscr{S}_{p,\leqslant A,\leqslant B;M}(f)$ is attained. If $A \leqslant BM^{p-2}$ (recall Remark 2.6), then the supremum $\mathscr{S}_{p,A,B;M}(f)$ is attained as well.

Proposition 2.8. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be any twice continuously differentiable function such that f is convex, f'' is strictly convex, and the condition (13) holds. Then $\mathscr{S}_{p,A,B;M}(f)$ is nondecreasing in $B \in [A/M^{p-2}, \infty)$ for each $A \in (0,\infty)$ and in $A \in (0, BM^{p-2}]$ for each $B \in (0,\infty)$; in particular, it follows that*

$$\mathscr{S}_{p,A,B;M}(f) = \mathscr{S}_{p,\leqslant A,\leqslant B;M}(f) \tag{47}$$

for any positive real A and B such that $A \leqslant BM^{p-2}$.

Take any $H_* \in \mathscr{H}_{p,A,B;M}$ such that

$$\mathsf{E}\,|X + Y_{H_*}|^p = \mathscr{S}_{p,A,B;M}(\mathsf{E}\,|X + \cdot|^p) = \mathscr{S}_{p,\leqslant A,\leqslant B;M}(\mathsf{E}\,|X + \cdot|^p); \tag{48}$$

according to Propositions 2.7 and 2.8, such a measure H_* exists.

Here and in what follows, the r.v.'s entering the same expression are assumed to be independent. Thus, the r.v.'s X and Y_{H_*} in (48) are assumed to be independent.

Proposition 2.9. *Suppose that $p > 5$. Then for the support set $\operatorname{supp}(H_*)$ of the extremal measure H_* one has*

$$\operatorname{card}\big((0,\infty) \cap \operatorname{supp}(H_*)\big) \leqslant 1 \quad \textit{and} \quad \operatorname{card}\big((-\infty,0) \cap \operatorname{supp}(H_*)\big) \leqslant 1, \tag{49}$$

where card *denotes the cardinality of the set.*

Proposition 2.10. *Suppose that $p > 4$. Then $H_*(\{0\}) = 0$.*

Proof of Proposition 2.7. Let us only show that $\mathscr{S}_{p,A,B;M}(f)$ is attained; that $\mathscr{S}_{p,\leqslant A,\leqslant B;M}$ is so is shown similarly and even a bit more easily. W.l.o.g. $X = 0$, since (46) holds for the function $\mathsf{E}\, f(X + \cdot)$ in place of f, with the constant factor $C_1 := C\,\mathsf{E}\, e^{c|X|} < \infty$ in place of C. Let (H_m) be a sequence in $\mathscr{H}_{p,A,B;M}$ such that $\mathsf{E}\, f(Y_{H_m}) \to \mathscr{S}_{p,A,B;M}(f)$. Because the interval $[-M, M]$ is compact and the functions 1 and $|\cdot|^{p-2}$ are continuous and bounded on $[-M, M]$, w.l.o.g. the sequence (H_m) converges weakly to some $H_* \in \mathscr{H}_{p,A,B;M}$, which implies that $Y_{H_m} \to Y_{H_*}$ in distribution, since the function e_2, defined in (45), is continuous

and bounded on any bounded subset of $\mathbb{C}$. Moreover, by the analytic extension of (44),

$$\mathsf{E}\cosh(kY_H) = \tfrac{1}{2}\big(\exp\big\{k^2 \textstyle\int e_2(kx)\,H(\mathrm{d}x)\big\} + \exp\big\{k^2 \textstyle\int e_2(-kx)\,H(\mathrm{d}x)\big\}\big) \leqslant \exp\{k^2\,e_2(|k|M)\,B\} < \infty$$

for all real k and all $H \in \mathscr{H}_{p,A,B;M}$, since the function e_2 is increasing on $\mathbb{R}$. So, by [2, Theorem 5.4] and in view of (46),

$$\mathscr{S}_{p,A,B;M}(f) = \lim_m \mathsf{E}\,f(Y_{H_m}) = \mathsf{E}\,f(Y_{H_*}). \tag{50}$$

□

Proof of Proposition 2.8. Let us only show that $\mathscr{S}_{p,A,B;M}(f)$ is nondecreasing in A and in B; then (47) follows immediately. As in the proof of Proposition 2.7, w.l.o.g. $X = 0$.

In accordance with (50), take any $H_* \in \mathscr{H}_{p,A,B;M}$ such that $\mathsf{E}\,f(Y_{H_*}) = \mathscr{S}_{p,A,B;M}(f)$. Then $H_t := H_* + t\delta_0 \in \mathscr{H}_{p,A,B+t;M}$ for all real $t \geqslant 0$, where δ_u denotes the Dirac probability measure at u. Also, $\mathsf{E}\,f(Y_{H_t}) = \mathsf{E}\,g(\sqrt{t}Z)$, where $g(u) := \mathsf{E}\,f(Y_{H_*} + u)$. So, by Lemma 2.3, the right derivative of $\mathsf{E}\,f(Y_{H_t})$ in t at $t = 0$ is $\frac{1}{2}\,g''(0) = \frac{1}{2}\,\mathsf{E}\,f''(Y_{H_*}) \geqslant 0$, since f is a twice differentiable convex function. Therefore, for the lower right derivative of $\mathscr{S}_{p,A,B;M}(f)$ in B one has

$$\liminf_{t\downarrow 0} \frac{\mathscr{S}_{p,A,B+t;M}(f) - \mathscr{S}_{p,A,B;M}(f)}{t} \geqslant \liminf_{t\downarrow 0} \frac{\mathsf{E}\,f(Y_{H_t}) - \mathsf{E}\,f(Y_{H_*})}{t} = \tfrac{1}{2}\,g''(0) \geqslant 0, \tag{51}$$

which shows that $\mathscr{S}_{p,A,B;M}(f)$ is nondecreasing in B.

To show that $\mathscr{S}_{p,A,B;M}(f)$ is nondecreasing in A, take any $A \in (0, BM^{p-2})$; cf. (39). Then

$$H_*\big((-M,M)\big) > 0, \tag{52}$$

because otherwise $\operatorname{supp}(H_*) \subseteq \{-M, M\}$ and hence $A = BM^{p-2}$.

Now consider the case when

$$H_*\big((-M,M)\setminus\{0\}\big) > 0,$$

so that there exists some $b \in \operatorname{supp}(H_*) \cap (-M,M) \setminus \{0\}$. Take now any $b_0 \in (0, |b|)$ and define the measures $\tilde{H}$ and G by the conditions

$$\tilde{H}(E) := H_*\big(E \cap (-b_0, b_0)\big) \quad \text{and} \quad G(E) := \int_{E\setminus(-b_0,b_0)} \frac{1}{x^2}\,H_*(\mathrm{d}x), \tag{53}$$

so that

$$H_*(E) := \tilde{H}(E) + \int_E x^2 G(\mathrm{d}x) \tag{54}$$

for all Borel sets $E \subseteq \mathbb{R}$. Then

$$\mathscr{S}_{p,A,B;M}(f) = \mathsf{E}\,f(Y_{H_*}) = \mathsf{E}\,f(Y_{\tilde{H}} + X_G). \tag{55}$$

Define the signed measure

$$\Delta := \Delta_\delta(E) := \Delta_1 - \Delta_2, \tag{56}$$

where

$$\Delta_1(E) := \Delta_{1;\delta}(E) := \frac{M+b}{2M^3}\,\mathbf{I}\{M \in E\} + \frac{M-b}{2M^3}\,\mathbf{I}\{-M \in E\}, \tag{57}$$

$$\Delta_2(E) := \Delta_{2;\delta}(E) := \frac{1}{G([b-\delta, b+\delta])}\int_{E\cap[b-\delta,b+\delta]} \frac{1}{x^2}\,G(\mathrm{d}x) \tag{58}$$

for any Borel set $E \subseteq \mathbb{R}$ and any $\delta \in (0,|b|)$, so that $G([b-\delta,b+\delta]) > 0$. Also, then the measure G_t as in (9) is nonnegative for all $t \in \big[0, t_0\big]$, where $t_0 := (|b|-\delta)^2 G\big([b-\delta,b+\delta]\big)$. So, by Lemma 2.2, one has (12) with

$$\tilde f(\cdot) := \mathsf{E}\, f(Y_{\tilde H} + \cdot) \tag{59}$$

in place of f in the definition (11) of g_t, so that here $g_t = g_{\tilde f;t}$ and

$$g = g_{\tilde f;0}.$$

So,

$$\begin{aligned}
\lim_{\delta\downarrow 0}\lim_{t\downarrow 0} & \frac{\mathsf{E}\,\tilde f(X_{G_t}) - \mathsf{E}\,\tilde f(X_G)}{t} \\
&= \lim_{\delta\downarrow 0}\int_{\mathbb{R}} [g(u) - g(0) - g'(0)u]\Delta_\delta(\mathrm{d}u) \\
&= \frac{M+b}{2M}\,\frac{g(M)-g(0)-g'(0)M}{M^2} + \frac{M-b}{2M}\,\frac{g(-M)-g(0)+g'(0)M}{M^2} \\
&\qquad - \frac{g(b)-g(0)-g'(0)b}{b^2} \\
&= \int_0^1 \Big(\frac{M+b}{2M} g''(Ms) + \frac{M-b}{2M} g''(-Ms) - g''(bs)\Big)(1-s)\mathrm{d}s > 0,
\end{aligned} \tag{60}$$

because f'' was assumed strictly convex and hence, by (59) and dominated convergence, so is $g''(\cdot) = \mathsf{E}\, f''(Y_{\tilde H} + X_G + \cdot)$.

Now (55), (59), and (60) imply that eventually

$$\begin{aligned}
\mathscr{S}_{p,A,B;M}(f) &= \mathsf{E}\, f(Y_{\tilde H} + X_G) \\
&= \mathsf{E}\,\tilde f(X_G) < \mathsf{E}\,\tilde f(X_{G_t}) \\
&= \mathsf{E}\, f(Y_{\tilde H} + X_{G_t}) = \mathsf{E}\, f(Y_{H_t}),
\end{aligned} \tag{61}$$

where

$$H_t(E) := \tilde H(E) + \int_E x^2 G_t(\mathrm{d}x) = H_*(E) + t\int_E x^2\Delta(\mathrm{d}x)$$

for all Borel sets $E \subseteq \mathbb{R}$; cf. (53) and (54). In this context, we say that an assertion $\mathscr{A} = \mathscr{A}_{\delta,t}$ holds "eventually" if $\exists \delta_0 \in (0,\infty)\ \forall \delta \in (0,\delta_0)\ \exists t_\delta \in (0,t_0)$

$\forall t \in \big(0, t_\delta)\big)$ $\mathscr{A}_{\delta,t}$ holds; recall here that, in view of the definitions (56), (57), (58), and (9), G_t depends not only on t but also on δ.

On the other hand, for all $t \in (0, t_0)$ one has $\int_{\mathbb{R}} H_t(\mathrm{d}x) = \int_{\mathbb{R}} H_*(\mathrm{d}x) + t\int_{\mathbb{R}} x^2\Delta(\mathrm{d}x) = B + t\int_{\mathbb{R}} x^2\Delta(\mathrm{d}x) = B$ and $\int_{\mathbb{R}} |x|^{p-2}H_t(\mathrm{d}x) = A + ta$ and hence $H_t \in \mathscr{H}(p, A+ta, B; M)$, where $a := \int_{\mathbb{R}} |x|^p\Delta(\mathrm{d}x) \geqslant \big(M^{p-2} - (|b|+\delta)^{p-2}\big) > 0$ for all small enough $\delta > 0$. So, by (42), eventually $\mathscr{S}(p, A+ta, B; M)(f) \geqslant \mathsf{E}\, f(Y_{H_t})$, whence, by (61), $\mathscr{S}(p, \cdot, B; M)(f)$ is increasing in a right neighborhood of the previously chosen value of $A \in (0, BM^{p-2})$ – in the case when $H_*\big((-M, M) \setminus \{0\}\big) > 0$; recall that, of course, H_* depends on A.

The case when $H_*\big((-M, M) \setminus \{0\}\big) = 0$ is considered quite similarly. In this case, the condition (78) must hold, in view of (52). So, here one can use

(i) 0 in place of b;
(ii) $\mathsf{E}\, f(Y_{\tilde{H}_0} + \cdot)$ in place of $\tilde{f}$ in (59), where $\tilde{H}_0$ is as in (79);
(iii) Lemma 2.4 $\big($with $\Delta = \Delta_1$, Δ_1 as in (57), and $\kappa = -1\big)$ in place of Lemma 2.2,

thus obtaining (60) with 0 in place of b and $\frac{1}{2}g''(0)$ in place of $\frac{g(b)-g(0)-g'(0)b}{b^2}$. We conclude that in the case $H_*\big((-M, M) \setminus \{0\}\big) = 0$ as well, $\mathscr{S}(p, \cdot, B; M)(f)$ is increasing in a right neighborhood of the value of $A \in (0, BM^{p-2})$.

Since A was chosen arbitrarily in the interval $(0, BM^{p-2})$, to complete the proof of Proposition 2.8, it remains to note that $\mathscr{S}(p, A, B; M)(f)$ is left-upper semi-continuous in A at $A = A_{\max} := BM^{p-2}$; that is,

$$\limsup_{A\uparrow A_{\max}} \mathscr{S}(p, A, B; M)(f) \leqslant \mathscr{S}(p, A_{\max}, B; M)(f).$$

Indeed, take any sequence (A_m) such that $A_m \uparrow A_{\max}$ and

$$\lim_{m\to\infty} \mathscr{S}(p, A_m, B; M)(f) > \mathscr{S}(p, A_{\max}, B; M)(f).$$

By Proposition 2.7, for each m there is some measure $H_m \in \mathscr{H}(p, A_m, B; M)$ such that $\mathsf{E}\, f(Y_{H_m}) = \mathscr{S}(p, A_m, B; M)(f)$. Passing to a subsequence of the sequence (A_m), w.l.o.g. one may assume that H_m converges weakly on the compact set $[-M, M]$ to some measure H_{**}. Since the functions 1, $|\cdot|^{p-2}$, and f are continuous, it follows that $H_{**} \in \mathscr{H}(p, A_{\max}, B; M)$ and $\mathscr{S}(p, A_m, B; M)(f) = \mathsf{E}\, f(Y_{H_m}) \longrightarrow \mathsf{E}\, f(Y_{H_{**}}) \leqslant \mathscr{S}(p, A_{\max}, B; M)(f)$ as $m \to \infty$, which contradicts the assumption on the sequence (A_m). □

Proof of Proposition 2.9. To obtain a contradiction, suppose that there exist b and b_1 such that $0 < b < b_1 < \infty$ and $\{b, b_1\} \subseteq \operatorname{supp}(H_*)$. In view of possible rescaling (i.e., replacing X, A, B, M, and $H_*(\mathrm{d}x)$ by X/b_1, A/b_1^p, B/b_1^2, M/b_1, and $H_*(b_1\,\mathrm{d}y)/b_1^2$, respectively), w.l.o.g. assume that $b_1 = 1$, so that

$$0 < b < 1.$$

Take any $b_0 \in (0, b)$ and let the measures $\tilde{H}$ and G be as in (53), so that (54) holds and, by Proposition 2.8,

$$\mathscr{S}_{p,\leqslant A,\leqslant B;M}(\mathsf{E}\,|X+\cdot|^p) = \mathsf{E}\,|X+Y_{H_*}|^p = \mathsf{E}\,|X+Y_{\tilde{H}}+X_G|^p. \tag{62}$$

Introduce now

$$k := \frac{b^2(1-b^{p-3})}{p-3}, \tag{63}$$

take any

$$a \in (0, 1/k), \tag{64}$$

and then also introduce

$$\varepsilon := a\big(b - b^{p-1} - (p-2)k\big), \quad \tilde{a} := 1 + a\big(b^{p-1} + (p-1)k\big), \quad \text{and} \quad \tilde{b} := 1 - ka. \tag{65}$$

Note that the condition (64) implies $\tilde{b} \in (0,1)$. Observe also that $\varepsilon = abr(b)/(p-3)$, where $r(b) := p - 3 - (p-2)b + b^{p-2}$, and $r(1) = 0$ and $r'(b) = -(p-2)(1-b^{p-3}) < 0$ for $b \in (0,1)$, so that $\varepsilon > 0$. Take now any $\beta \in (0, \infty)$ and let

$$\alpha := \varepsilon/\beta, \tag{66}$$

so that $\alpha > 0$.

Define the signed measure

$$\Delta := \Delta_{a,\beta,\delta} := \Delta_1 - \Delta_2, \tag{67}$$

where

$$\Delta_1(E) := \Delta_{1;a,\beta,\delta}(E) := \frac{\alpha}{\beta}\,\mathbf{I}\{\beta \in E\} + \frac{\tilde{a}}{\tilde{b}}\,\mathbf{I}\{\tilde{b} \in E\}, \tag{68}$$

$$\Delta_2(E) := \Delta_{2;a,\beta,\delta}(E) := \frac{a}{b}\,\frac{G(E \cap [b-\delta, b+\delta])}{G([b-\delta, b+\delta])} + \frac{G(E \cap [1-\delta, 1+\delta])}{G([1-\delta, 1+\delta])} \tag{69}$$

for any Borel set $E \subseteq \mathbb{R}$, and δ is any real number in the interval $(0, \frac{1-b}{2})$, so that the denominators $G([b-\delta, b+\delta])$ and $G([1-\delta, 1+\delta])$ are strictly positive and the intervals $[b-\delta, b+\delta]$ and $[1-\delta, 1+\delta]$ are disjoint, in view of the assumptions $\{b, b_1\} \subseteq \operatorname{supp}(H_*)$ and $b_1 = 1$. Define here the measure G_t as in (9), but with Δ as in (67). This measure is nonnegative for all $t \in [0, t_0]$, where

$$t_0 := \min\big(\tfrac{b}{a} G([b-\delta, b+\delta]), G([1-\delta, 1+\delta])\big) > 0.$$

So, by Lemma 2.2, one has (12) with

$$f_p(\cdot) := \mathsf{E}\,|X + Y_{\tilde{H}} + \cdot|^p \tag{70}$$

in place of f in the definition (11) of g_t, so that here $g_t = g_{f_p;t}$ and

$$g = g_{f_p;0}. \tag{71}$$

Let now $\delta \downarrow 0$ and $\beta \downarrow 0$. Then, in view of (66), one has

$$\frac{\alpha}{\beta}\big(g(\beta) - g(0) - \beta g'(0)\big) \to \frac{\varepsilon}{2}\, g''(0)$$

and hence

$$\begin{aligned}\lim_{\beta,\delta\downarrow 0}\lim_{t\downarrow 0}&\frac{\mathsf{E}\,f_p(X_{G_t})-\mathsf{E}\,f_p(X_G)}{t}\\ &=\lim_{\beta,\delta\downarrow 0}\int_{\mathbb{R}}[g(u)-g(0)-g'(0)u]\Delta_{a,\beta,\delta}(\mathrm{d}u)\\ &=\frac{\varepsilon}{2}\,g''(0)+\tilde a\tilde b\,\frac{g(\tilde b)-g(0)-\tilde b g'(0)}{\tilde b^2}\\ &\qquad-ab\,\frac{g(b)-g(0)-bg'(0)}{b^2}-\big(g(1)-g(0)-g'(0)\big).\end{aligned}$$

Letting further $a\downarrow 0$ and recalling the definitions (65) and (63) of $\varepsilon,\tilde a,\tilde b,k$, one obtains

$$\begin{aligned}\mathscr{L}&:=\frac{1}{b^2}\lim_{a\downarrow 0}\lim_{\beta,\delta\downarrow 0}\lim_{t\downarrow 0}\frac{\mathsf{E}\,f_p(X_{G_t})-\mathsf{E}\,f_p(X_G)}{at}\\ &=\frac{p-3b^{p-3}}{p-3}R_2(g;1)-\frac{1-b^{p-3}}{p-3}R_1(g';1)-\frac{1}{b^3}R_2(g;b),\end{aligned}\tag{72}$$

where

$$R_j(g;x):=g(x)-\sum_{i=0}^{j}\frac{x^i}{i!}\,g^{(i)}(0)=\frac{x^{j+1}}{j!}\int_0^1(1-s)^j\,g^{(j+1)}(xs)\mathrm{d}s,$$

the jth Maclaurin remainder for g at point x. It follows that

$$2\mathscr{L}=\int_0^1(1-s)F(p,b,s)\mathrm{d}s,$$

where

$$\begin{aligned}F(p,b,s)&:=(1-s)[g'''(s)-g'''(bs)]+(1-3s)g'''(s)\frac{1-b^{p-3}}{p-3}\\ &=(1-s)s\int_b^1 g^{(4)}(su)\mathrm{d}u+(1-3s)g'''(s)\int_b^1 u^{p-4}\mathrm{d}u.\end{aligned}$$

Also, integration by parts gives $\int_0^1(1-s)(1-3s)g'''(s)\mathrm{d}s=-\int_0^1(1-s)^2 sg^{(4)}(s)\mathrm{d}s$. So,

$$2\mathscr{L}=\int_0^1\mathrm{d}s\int_b^1\mathrm{d}u\,(1-s)^2 s\,[g^{(4)}(su)-u^{p-4}g^{(4)}(s)].\tag{73}$$

By (71), (70), (11), and (54),

$$g(x)=\mathsf{E}\,|W+x|^p,$$

where $W:=X+Y_{\tilde H}+X_G=X+Y_{H_*}$; note that, in view of (3) and (44), $\mathsf{E}\,W=0$ and $\mathsf{E}\,W^2>0$, where the latter inequality follows because $\operatorname{card}\operatorname{supp}(H_*)\geqslant 2$ and hence $H_*\neq 0$ and thus the r.v. Y_{H_*} is non-degenerate. So,

$$\frac{g^{(4)}(su)-u^{p-4}g^{(4)}(s)}{p(p-1)(p-2)(p-3)}=\mathsf{E}\,|W+su|^{p-4}-\mathsf{E}\,|uW+su|^{p-4}>0\tag{74}$$

for all $u \in (0,1)$ and $s > 0$. Indeed, introducing $\psi(u) := \mathsf{E}\,|uW+v|^{p-4}$ for $u \in \mathbb{R}$ and $v > 0$, and also recalling the condition $p > 5$, one see that the function ψ is convex, with $\psi'(0) = 0$ and $\psi''(0) = (p-4)(p-5)\,\mathsf{E}\,W^2\,v^{p-6} > 0$. This implies that $\psi(u)$ is strictly increasing in $u \geqslant 0$, whence the inequality in (74) follows.

Thus, by (73), $\mathscr{L} > 0$. Now (62), (70), and (72) imply that eventually

$$\begin{aligned}\mathscr{S}_{p,\leqslant A,\leqslant B;M}(\mathsf{E}\,|X+\cdot|^p) &= \mathsf{E}\,|X+Y_{\tilde H}+X_G|^p \\ &= \mathsf{E}\,f_p(X_G) < \mathsf{E}\,f_p(X_{G_t}) \\ &= \mathsf{E}\,|X+Y_{\tilde H}+X_{G_t}|^p = \mathsf{E}\,|X+Y_{H_t}|^p,\end{aligned}$$

where

$$H_t(E) := \tilde H(E) + \int_E x^2 G_t(\mathrm{d}x) = H_*(E) + t\int_E x^2\Delta(\mathrm{d}x) \tag{75}$$

for all Borel sets $E \subseteq \mathbb{R}$; cf. (53) and (54). In this context, we say that an assertion $\mathscr{A} = \mathscr{A}_{a,\beta,\delta,t}$ holds "eventually" if $\exists a_0 \in (0,\infty)$ $\forall a \in (0,a_0)$ $\exists \beta_a \in (0,\infty)$ $\exists \delta_a \in (0,\infty)$ $\forall \beta \in (0,\beta_a)$ $\forall \delta \in (0,\delta_a)$ $\exists t_{\alpha,\beta} \in (0,\infty)$ $\forall t \in (0,t_{\alpha,\beta})$ $\mathscr{A}_{a,\beta,\delta,t}$ holds; recall here that, in view of the definitions (67), (68), (69), and (9), G_t depends, not only on t, but also on a,β,δ.

Thus, we obtain a contradiction with the definition of $\mathscr{S}_{p,\leqslant A,\leqslant B;M}$ in (43), because, as we shall check in moment, $H_t \in \mathscr{H}_{p,\leqslant A,\leqslant B;M}$ eventually. Indeed, by (68), (66), (65), and (69),

$$\begin{aligned}\int_{\mathbb{R}} x^2\Delta_1(\mathrm{d}x) &= \alpha\beta + \tilde a\tilde b = \varepsilon + \tilde a\tilde b \\ &= a\big(b - b^{p-1} - (p-2)k\big) + (1-ka)\big(1 + a[b^{p-1} + (p-1)k]\big) \\ &< ab - a\big(b^{p-1} + (p-2)k\big) + 1 + [b^{p-1} + (p-2)k]a = ab+1 = \lim_{\delta\downarrow 0}\int_{\mathbb{R}} x^2\Delta_2(\mathrm{d}x),\end{aligned}$$

so that eventually

$$\int_{\mathbb{R}} H_t(\mathrm{d}x) = \int_{\mathbb{R}} H_*(\mathrm{d}x) + t\int_{\mathbb{R}} x^2\Delta(\mathrm{d}x) < \int_{\mathbb{R}} H_*(\mathrm{d}x) \leqslant B,$$

by (75), (67), and (38).

Next, by (66),

$$\int_{\mathbb{R}} |x|^p\Delta_1(\mathrm{d}x) = \alpha\beta^{p-1} + \tilde a\tilde b^{p-1} \underset{\beta\downarrow 0}{\longrightarrow} \tilde a\tilde b^{p-1} < ab^{p-1} + 1 = \lim_{\delta\downarrow 0}\int_{\mathbb{R}} |x|^p\Delta_2(\mathrm{d}x), \tag{76}$$

where the inequality holds eventually, for all small enough $a > 0$. Indeed, in view of (65), this inequality can be rewritten as

$$f_\gamma(u) := [1 + (\gamma + r)u](1-u)^r - (1+\gamma u) < 0, \tag{77}$$

with $r := p-1 > 0$, $u := ka$, and $\gamma := b^r/k \geqslant 0$. Note that eventually $u \in (0,1)$. To verify inequality (77) for such u, note that $f_\gamma(u)$ decreases in γ, so that

w.l.o.g. $\gamma = 0$. The inequality $f_0(u) < 0$ is equivalent to $\ln(1+ru)+r\ln(1-u) < 0$, which is easy to check for $u \in (0,1)$ by differentiation. Now (76) implies that eventually

$$\int_{\mathbb{R}} |x|^{p-2} H_t(\mathrm{d}x) = \int_{\mathbb{R}} |x|^{p-2} H_*(\mathrm{d}x) + t\int_{\mathbb{R}} |x|^p \Delta(\mathrm{d}x) < \int_{\mathbb{R}} |x|^{p-2} H_*(\mathrm{d}x) \leqslant A,$$

again by by (75), (67), and (38).

Also, the conditions $H_* \in \mathscr{H}_{p,\leqslant A,\leqslant B;M}$, $\operatorname{supp} H \subseteq [-M, M]$ for all $H \in \mathscr{H}_{p,\leqslant A,\leqslant B;M}$, $\{b, b_1\} \subseteq \operatorname{supp}(H_*)$, and $b_1 = 1$ imply $M \geqslant 1$. So, $\operatorname{supp}(H_t) \subseteq \operatorname{supp}(H_*) \cup \{\beta, \tilde b\} \subseteq [-M, M]$ for all small enough β, in view of (65).

By (38), we conclude that indeed $H_t \in \mathscr{H}_{p,\leqslant A,\leqslant B;M}$ eventually. Thus, indeed the assumption that there exist b and b_1 such that $0 < b < b_1 < \infty$ and $\{b, b_1\} \subseteq \operatorname{supp}(H_*)$ leads to a contradiction, which proves the first inequality in (49). The second inequality there can be proved quite similarly or, alternatively, quickly obtained from the first one by a reflection. □

Proof of Proposition 2.10. The proof is somewhat similar to that Proposition 2.9. Suppose that, to the contrary,

$$\sigma := \sqrt{H_*(0)} > 0. \tag{78}$$

As noted in the proof of Proposition 2.8, necessarily $\operatorname{supp}(H_*) \setminus \{0\} \neq \emptyset$. So, in view of possible rescaling and reflection, w.l.o.g.

$$1 \in \operatorname{supp}(H_*).$$

Take any $b_0 \in (0,1)$ and let the measures $\tilde H$ and G be as in (53), so that (54) and (62) hold. Consider also the measure $\tilde H_0$ defined by the condition

$$\tilde H_0(E) := \tilde H(E \setminus \{0\}) \tag{79}$$

for all Borel sets $E \subseteq \mathbb{R}$, so that, by (48) and (78),

$$\mathscr{S}_{p,\leqslant A,\leqslant B;M}(\mathsf{E}\,|X + \cdot|^p) = \mathsf{E}\,|X + Y_{H_*}|^p = \mathsf{E}\,|X + Y_{\tilde H_0} + X_G + \sigma Z|^p. \tag{80}$$

Take next any $\beta \in \big(0, (\frac{p-2}{p-1})^{1/(p-2)}\big)$, so that

$$\varepsilon := \frac{\beta^{p-2}}{p-2} \in (0, \tfrac{1}{p-1}) \subset (0,1).$$

Introduce then

$$\alpha := \frac{1}{\beta}, \quad \tilde a := \frac{1-(p-1)\varepsilon}{(1-\varepsilon)^p}, \quad \text{and} \quad \tilde b := 1 - \varepsilon. \tag{81}$$

Define the signed measure

$$\Delta := \Delta_{\beta,\delta} := \Delta_1 - \Delta_2, \tag{82}$$

where

$$\Delta_1(E) := \Delta_{1;\beta,\delta}(E) := \frac{\alpha}{2\beta}\,\mathbf{I}\{\beta \in E\} + \frac{\alpha}{2\beta}\,\mathbf{I}\{-\beta \in E\} + \frac{\tilde a}{\tilde b}\,\mathbf{I}\{\tilde b \in E\}, \quad (83)$$

$$\Delta_2(E) := \Delta_{2;\beta,\delta}(E) := \frac{G(E \cap [1-\delta, 1+\delta])}{G([1-\delta,1+\delta])} \quad (84)$$

for any Borel set $E \subseteq \mathbb{R}$, and δ is any positive real number, so that $G([1-\delta, 1+\delta]) > 0$. Take $\kappa = -1$ and σ as in (78), and then take any

$$t_0 \in \big(0, \sigma^2 \wedge G([1-\delta, 1+\delta]\big).$$

Then for all $t \in [0, t_0]$ the measure G_t as in (9) but with Δ as in (82) is nonnegative and the condition $\kappa \in (-\sigma^2/t_0, \infty)$ in Lemma 2.4 holds. Recall now (79) and let

$$f_{p;0}(\cdot) := \mathsf{E}\,|X + Y_{\tilde H_0} + \cdot|^p; \quad (85)$$

cf. (70). Then, using $f_{p;0}$ in place of f in Lemma 2.4, one has

$$h_{f_{p;0};0} = g, \quad (86)$$

where g is as in (71). It follows by Lemma 2.4 that

$$\begin{aligned}
\mathscr{L}(\beta) := \lim_{\delta\downarrow 0}\lim_{t\downarrow 0} & \frac{\mathsf{E}\,f_{p;0}(X_{G_t} + Z\sqrt{\sigma^2 - t}) - \mathsf{E}\,f_{p;0}(X_G + Z\sigma)}{t} \\
&= -\frac12\, g''(0) + \lim_{\delta\downarrow 0}\int_{\mathbb{R}} [g(u) - g(0) - g'(0)u]\Delta_{\beta,\delta}(\mathrm{d}u) \\
= -\frac12\, g''(0) &+ \frac12\,\frac{g(\beta) - g(0) - \beta g'(0)}{\beta^2} + \frac12\,\frac{g(-\beta) - g(0) + \beta g'(0)}{\beta^2} \\
&\qquad + \tilde a \tilde b\,\frac{g(\tilde b) - g(0) - \tilde b g'(0)}{\tilde b^2} - \big(g(1) - g(0) - g'(0)\big).
\end{aligned}$$

Let now $\beta \downarrow 0$. Then, in view of (81) and because g is differentiable, the difference in the second line of the last two-line expression of $\mathscr{L}(\beta)$ is $O(\varepsilon) = O(\beta^{p-2}) = o(\beta^2)$. The rest of that two-line expression of $\mathscr{L}(\beta)$ can rewritten as

$$\frac{\beta^2}{2}\int_0^1 s^2(1-s)\mathrm{d}s \int_{-1}^1 g^{(4)}(sv\beta)(1-|v|)\mathrm{d}v,$$

whence

$$\begin{aligned}
\lim_{\beta\downarrow 0}\lim_{\delta\downarrow 0}\lim_{t\downarrow 0} & \frac{\mathsf{E}\,f_{p;0}(X_{G_t} + Z\sqrt{\sigma^2 - t}) - \mathsf{E}\,f_{p;0}(X_G + Z\sigma)}{\beta^2 t/24} \\
&= g^{(4)}(0) = p(p-1)(p-2)(p-3)\,\mathsf{E}\,|W|^{p-4},
\end{aligned}$$

where $W = X + Y_{\tilde H_0} + X_G + Z\sigma$, which is clearly a non-degenerate r.v., so that $\mathsf{E}\,|W|^{p-4} > 0$.

Now (62) and (85) imply that eventually

$$\begin{aligned}\mathscr{S}_{p,\leqslant A,\leqslant B;M}(\mathsf{E}\,|X+\cdot|^p)&=\mathsf{E}\,|X+Y_{\tilde H}+X_G|^p=\mathsf{E}\,|X+Y_{\tilde H_0}+X_G+\sigma Z|^p\\&=\mathsf{E}\,f_{p;0}(X_G+\sigma Z)<\mathsf{E}\,f_{p;0}(X_{G_t}+Z\sqrt{\sigma^2-t})\\&=\mathsf{E}\,|X+Y_{\tilde H_0}+X_{G_t}+Z\sqrt{\sigma^2-t}|^p=\mathsf{E}\,|X+Y_{H_t}|^p,\end{aligned}$$

where

$$\begin{aligned}H_t(E)&:=\tilde H_0(E)+(\sigma^2-t)\,\mathbf{I}\{0\in E\}+\int_E x^2G_t(\mathrm{d}x)\\&=H_*(E)-t\,\mathbf{I}\{0\in E\}+t\int_E x^2\Delta(\mathrm{d}x)\end{aligned}\tag{87}$$

for all Borel sets $E\subseteq\mathbb{R}$. In this context, we say that an assertion $\mathscr{A}=\mathscr{A}_{\beta,\delta,t}$ holds "eventually" if $\exists\beta_0\in\big(0,(\frac{p-2}{p-1})^{1/(p-2)}\big)$ $\forall\beta\in(0,\beta_0)$ $\exists\delta_\beta\in(0,\infty)$ $\forall\delta\in(0,\delta_\beta)$ $\exists t_\delta\in(0,t_0)$ $\forall t\in(0,t_\delta)$ $\mathscr{A}_{\beta,\delta,t}$ holds.

Thus, we obtain a contradiction with the definition of $\mathscr{S}_{p,\leqslant A,\leqslant B;M}$ in (43), because, as we shall check in moment, $H_t\in\mathscr{H}_{p,\leqslant A,\leqslant B;M}$ eventually. Indeed, by (82), (83), (84), and (81),

$$\lim_{\delta\downarrow0}\int_{\mathbb{R}}x^2\Delta(\mathrm{d}x)=\alpha\beta+\tilde a\tilde b-1=\frac{1-(p-1)\varepsilon}{(1-\varepsilon)^{p-1}}<1,$$

so that eventually

$$\int_{\mathbb{R}}H_t(\mathrm{d}x)=\int_{\mathbb{R}}H_*(\mathrm{d}x)-t+t\int_{\mathbb{R}}x^2\Delta(\mathrm{d}x)<\int_{\mathbb{R}}H_*(\mathrm{d}x)\leqslant B,$$

by (87) and (38).

Next, by (81),

$$\begin{aligned}\lim_{\delta\downarrow0}\int_{\mathbb{R}}|x|^p\Delta(\mathrm{d}x)&=\alpha\beta^{p-1}+\tilde a\tilde b^{p-1}-1\\&=(p-2)\varepsilon+\frac{1-(p-1)\varepsilon}{1-\varepsilon}-1=-\frac{(p-2)\varepsilon^2}{1-\varepsilon}<0.\end{aligned}\tag{88}$$

It follows that that eventually

$$\int_{\mathbb{R}}|x|^{p-2}H_t(\mathrm{d}x)=\int_{\mathbb{R}}|x|^{p-2}H_*(\mathrm{d}x)+t\int_{\mathbb{R}}|x|^p\Delta(\mathrm{d}x)<\int_{\mathbb{R}}|x|^{p-2}H_*(\mathrm{d}x)\leqslant A,\tag{89}$$

again by by (87) and (38).

Also, the conditions $H_*\in\mathscr{H}_{p,\leqslant A,\leqslant B;M}$, $\operatorname{supp}H\subseteq[-M,M]$ for all $H\in\mathscr{H}_{p,\leqslant A,\leqslant B;M}$, and $1\in\operatorname{supp}(H_*)$ imply $M\geqslant1$. So, $\operatorname{supp}(H_t)\subseteq\operatorname{supp}(H_*)\cup\{\beta,-\beta,\tilde b\}\subseteq[-M,M]$ in view of (81) for all small enough β.

By (38), we conclude that indeed $H_t\in\mathscr{H}_{p,\leqslant A,\leqslant B;M}$ eventually. Thus, the assumption (78) leads to a contradiction. □

2.5. Conclusion of the proof of Theorem 1.1

Take any
$$M > (A/B)^{1/(p-2)}. \tag{90}$$
Denote the last supremum in (4) by RHS_*, and then let $\mathrm{RHS}_{p,M}$ and $\mathrm{RHS}^\circ_{p,M}$ denote the similar suprema with the condition $(c_1, c_2, \lambda_1, \lambda_2) \in (0,\infty)^4$ replaced by $(c_1, c_2, \lambda_1, \lambda_2) \in [0,M]^2 \times (0,\infty)^2$ and $(c_1, c_2, \lambda_1, \lambda_2) \in (0,M)^2 \times (0,\infty)^2$, respectively. In fact, by (3) and dominated convergence, it is easy to see that $\mathsf{E}\,|X + c_1\tilde{\Pi}_{\lambda_1} - c_2\tilde{\Pi}_{\lambda_2}|^p$ is continuous in $(c_1, c_2, \lambda_1, \lambda_2) \in [0,\infty)^2 \times (0,\infty)^2$ and hence
$$\mathrm{RHS}_{p,M} = \mathrm{RHS}^\circ_{p,M}\,. \tag{91}$$

Consider now the case $p > 5$. Note that any $H_* \in \mathscr{H}_{p,A,B;M}$ satisfying the conditions (49) and $H_*(\{0\}) = 0$ in Propositions 2.9 and 2.10 can be represented as $c_1^2\lambda_1\delta_{c_1} + c_2^2\lambda_2\delta_{-c_2}$ for some $(c_1, c_2, \lambda_1, \lambda_2) \in [0,M]^2 \times (0,\infty)^2$ such that
$$c_1^2\lambda_1 + c_2^2\lambda_2 = B \text{ and } c_1^p\lambda_1 + c_2^p\lambda_2 = A. \tag{92}$$

Moreover, then, by (44),
$$Y_{H_*} \overset{\mathrm{D}}{=} c_1\tilde{\Pi}_{\lambda_1} - c_2\tilde{\Pi}_{\lambda_2},$$
where $\overset{\mathrm{D}}{=}$ denotes the equality in distribution. So, by Propositions 2.8, 2.9, 2.10, and identity (91),
$$\mathscr{S}_{p,\leqslant A,\leqslant B;M}(\mathsf{E}\,|X + \cdot|^p) = \mathscr{S}_{p,A,B;M}(\mathsf{E}\,|X + \cdot|^p) \leqslant \mathrm{RHS}^\circ_{p,M} \tag{93}$$
for any real $p > 5$. In fact, one can write the equality in place of the inequality in (93). However, this equality will not be needed in the current proof. Rather, note that (93) holds for $p = 5$ as well. Take any $H \in \mathscr{H}_{5,A,B;M}$ and any sequence (p_n) in $(5,\infty)$ such that $p_n \downarrow 5$ as $n \to \infty$. Then $|x|^{p_n-2} \to |x|^{5-2}$ uniformly in $x \in [-M,M]$ and hence
$$A_n := \int_{\mathbb{R}} |x|^{p_n-2}H(\mathrm{d}x) \to \int_{\mathbb{R}} |x|^{5-2}H(\mathrm{d}x) = A.$$
So, by the Fatou lemma, the definition (42) of $\mathscr{S}_{p,A,B;M}(f)$, and Proposition 2.7,
$$\begin{aligned}\mathsf{E}\,|X + Y_H|^5 \leqslant \liminf_n \mathsf{E}\,|X + Y_H|^{p_n} &\leqslant \liminf_n \mathscr{S}_{p_n,A_n,B;M}(\mathsf{E}\,|X + \cdot|^{p_n}) \\ &= \liminf_n \mathsf{E}\,|X + Y_{H_{*,n}}|^{p_n},\end{aligned} \tag{94}$$
where $H_{*,n}$ is any measure in $\mathscr{H}_{p_n,A_n,B;M}$ such that
$$\mathsf{E}\,|X + Y_{H_{*,n}}|^{p_n} = \mathscr{S}_{p_n,A_n,B;M}(\mathsf{E}\,|X + \cdot|^{p_n})$$
– cf. (48). By a compactness argument, w.l.o.g. $H_{*,n} \to H_{**}$ weakly for some measure $H_{**} \in \mathscr{H}_{5,A,B;M}$. Also (cf. (50)), $\mathsf{E}\,|X + Y_{H_{*,n}}|^{p_n} \to \mathsf{E}\,|X + Y_{H_{**}}|^5$,

and so, by (94), $\mathsf{E}\,|X+Y_H|^5 \leqslant \mathsf{E}\,|X+Y_{H_{**}}|^5$, for any $H \in \mathscr{H}_{5,A,B;M}$; that is, the supremum $\mathscr{S}_{5,A,B;M}(\mathsf{E}\,|X+\cdot|^5)$ is attained at H_{**}. So, by Proposition 2.10 (which holds for all $p>4$), $H_{**}(\{0\})=0$. Also, Proposition 2.9 (with p_n and A_n in place of p and A) and the weak convergence $H_{*,n} \to H_{**}$ imply that $\operatorname{card}\big((0,\infty)\cap\operatorname{supp}(H_{**})\big) \leqslant 1$ and $\operatorname{card}\big((-\infty,0)\cap\operatorname{supp}(H_{**})\big) \leqslant 1$. Therefore and view of (91), one concludes that (93) holds indeed for $p=5$ as well.

Take any $\mathbf{X}=(X_1,\dots,X_n) \in \mathscr{X}_{p;X;\leqslant A,\leqslant B}$. By Proposition 2.1, for each $i \in \{1,\dots,n\}$ and each real $M>0$ there is a truncated version $X_{i,M}$ of X_i such that

(i) $\mathsf{E}\,X_{i,M}=0$;
(ii) $|X_{i,M}| \leqslant M \wedge |X_i|$;
(iii) $X_{i,M} \to X_i$ a.s. as $M \to \infty$;
(iv) $X_{1,M},\dots,X_{n,M}$ are independent.

Then obviously

$$(X_{1,M},\dots,X_{n,M}) \in \mathscr{X}_{p;X;\leqslant A,\leqslant B}. \tag{95}$$

Letting now $S_M := X_{1,M}+\dots+X_{n,M}$, one also has $|X+S_M|^p \leqslant n^{p-1}\big(|X|^p + \sum_1^n |X_{i,M}|^p\big) \leqslant n^{p-1}\big(|X|^p+\sum_1^n |X_i|^p\big)$. So, by dominated convergence,

$$\mathsf{E}\,|X+S_M|^p \underset{M\to\infty}{\longrightarrow} \mathsf{E}\,|X+S|^p. \tag{96}$$

On the other hand, by Theorem A $\big($with $\mathsf{E}\,|X+\cdot|^p$ and $X_{i,M}$ in place of f and $X_i\big)$, (7), and (44),

$$\mathsf{E}\,|X+S_M|^p \leqslant \mathsf{E}\,|X+Y_{H_M}|^p, \tag{97}$$

where

$$H_M(E) := \int_E x^2 \sum_1^n \mathsf{P}(X_{i,M} \in \mathrm{d}x)$$

for all Borel sets $E \subseteq \mathbb{R}$. It follows from (95) that the measure H_M is in $\mathscr{H}_{p,\leqslant A,\leqslant B;M}$. So, by (97) and (93), $\mathsf{E}\,|X+S_M|^p \leqslant \mathrm{RHS}^\circ_{p,M} \leqslant \mathrm{RHS}_*$, for each M as in (90). Now (96) yields

$$\mathsf{E}\,|X+S|^p \leqslant \mathrm{RHS}_*\,. \tag{98}$$

Thus, the first supremum in (4) is no greater than the third supremum there, RHS_*.

To complete the proof of Theorem 1.1, it remains to show that the second supremum in (4) is no less than RHS_*. This can be done in a rather standard manner; cf. e.g. [14, 15] or the paragraphs containing formulas (6.1) and (6.2) in [8]. Take any $p \geqslant 5$. Take any $(c_1,c_2,\lambda_1,\lambda_2) \in (0,\infty)^4$ such that

$$c_1^2\lambda_1 + c_2^2\lambda_2 = B \text{ and } c_1^p\lambda_1 + c_2^p\lambda_2 = A. \tag{99}$$

For each natural n and all $j \in \{1,\dots,n\}$, let

$$Z_{j,n} := W_{j,n} - \mathsf{E}\,W_{j,n}, \tag{100}$$

where the $W_{j,n}$'s are i.i.d. r.v.'s which are independent of X and have the characteristic function

$$t \longmapsto 1 + \kappa_n\big(\tfrac{\lambda_1}{n}\,(e^{i\gamma_n c_1 t} - 1) + \tfrac{\lambda_2}{n}\,(e^{-i\gamma_n c_2 t} - 1)\big), \tag{101}$$

where in turn κ_n and γ_n are positive real numbers, which will be specified a bit later; at this point, let us only assume that $\kappa_n < n/(\lambda_1 + \lambda_2)$ for all n – so that (101) indeed defines a characteristic function. Then

$$\sum_1^n \mathsf{E}\,|Z_{j,n}|^r = n\,\mathsf{E}\,|Z_{1,n}|^r = F_r\big(\tfrac{1}{n}, \kappa_n, \gamma_n\big),$$

where

$$\begin{aligned} F_r(\alpha,\kappa,\gamma) :=& \kappa\gamma^r \sum_{k=1}^{2} \lambda_k \big|c_k - (c_1\lambda_1 - c_2\lambda_2)\kappa\alpha\big|^r \\ &+ \big(1 - (\lambda_1+\lambda_2)\kappa\alpha\big)(\kappa\gamma)^r\,|c_1\lambda_1 - c_2\lambda_2|^r\,|\alpha|^{r-1}\operatorname{sign}\alpha. \end{aligned}$$

Introducing now the vector function $\mathbf{F} := (F_2, F_p)$, we see that it is continuously differentiable on $\mathbb{R}\times(0,\infty)^2$ and the Jacobian matrix $\begin{pmatrix} \frac{\partial F_2}{\partial\kappa} & \frac{\partial F_2}{\partial\gamma} \\ \frac{\partial F_p}{\partial\kappa} & \frac{\partial F_p}{\partial\gamma} \end{pmatrix}$ at the point $(\alpha,\kappa,\gamma) = (0,1,1)$ is $\begin{pmatrix} B & 2B \\ A & pA \end{pmatrix}$, which is nonsingular; here we took (99) into account. Moreover, again by (99), $\mathbf{F}(0,1,1) = (B,A)$. So, by the implicit function theorem, there exist a positive real number α_0 and continuously differentiable functions $\tilde\kappa\colon(-\alpha_0,\alpha_0)\to\mathbb{R}$ and $\tilde\gamma\colon(-\alpha_0,\alpha_0)\to\mathbb{R}$ such that $\tilde\kappa(0) = \tilde\gamma(0) = 1$ and $\mathbf{F}\big(\alpha,\tilde\kappa(\alpha),\tilde\gamma(\alpha)\big) = (B,A)$ for all $\alpha\in(-\alpha_0,\alpha_0)$. For all natural $n > 1/\alpha_0$, letting now

$$\kappa_n := \tilde\kappa(\tfrac{1}{n}) \quad\text{and}\quad \gamma_n := \tilde\gamma(\tfrac{1}{n}),$$

one sees that $\sum_1^n \mathsf{E}\,|Z_{j,n}|^2 = B$ and $\sum_1^n \mathsf{E}\,|Z_{j,n}|^p = A$, so that

$$\mathbf{Z}_n := (Z_{1,n},\dots,Z_{n,n}) \in \mathscr{X}_{p;X;A,B}.$$

Moreover, $\kappa_n \to \tilde\kappa(0) = 1$ and $\gamma_n \to \tilde\gamma(0) = 1$ (the convergence in this context is of course as $n\to\infty$). So, by (100) and (101),

$$\begin{aligned} &\mathsf{E}\exp\big(itS_{\mathbf{Z}_n}\big) \\ &= \big[1 + \kappa_n\big(\tfrac{\lambda_1}{n}\,(e^{i\gamma_n c_1 t} - 1) + \tfrac{\lambda_2}{n}\,(e^{-i\gamma_n c_2 t} - 1)\big)\big]^n \exp\big(-i\kappa_n\gamma_n(c_1\lambda_1 - c_2\lambda_2)t\big) \\ &\longrightarrow \exp\big(\lambda_1\,(e^{ic_1 t} - 1 - ic_1 t) + \lambda_2\,(e^{-ic_2 t} - 1 + ic_2 t)\big) = \mathsf{E}\exp\big(i(c_1\tilde\Pi_{\lambda_1} - c_2\tilde\Pi_{\lambda_2})t\big) \end{aligned}$$

for all real t, so that $S_{\mathbf{Z}_n} \to c_1\tilde\Pi_{\lambda_1} - c_2\tilde\Pi_{\lambda_2}$ in distribution. Now, by the Fatou lemma for the convergence in distribution [2, Theorem 5.3],

$$\liminf_n \mathsf{E}\,|X + S_{\mathbf{Z}_n}|^p \geqslant \mathsf{E}\,|X + c_1\tilde\Pi_{\lambda_1} - c_2\tilde\Pi_{\lambda_2}|^p,$$

so that indeed the second supremum in (4) is no less than the third one there, RHS_*. $\square$

References

[1] E. V. Bestsennaya and S. A. Utev. An exact upper bound for the even moment of sums of independent random variables. *Sibirsk. Mat. Zh.*, 32(1):171–173, 222, 1991.

[2] P. Billingsley. *Convergence of probability measures.* John Wiley & Sons Inc., New York, 1968.

[3] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973.

[4] T. Figiel, P. Hitczenko, W. B. Johnson, G. Schechtman, and J. Zinn. Extremal properties of Rademacher functions with applications to the Khintchine and Rosenthal inequalities. *Trans. Amer. Math. Soc.*, 349(3):997–1027, 1997.

[5] R. Ibragimov and S. Sharakhmetov. On an exact constant for the Rosenthal inequality. *Teor. Veroyatnost. i Primenen.*, 42(2):341–350, 1997.

[6] R. Ibragimov and S. Sharakhmetov. On extremal problems and best constants in moment inequalities. *Sankhyā Ser. A*, 64(1):42–56, 2002.

[7] I. Pinelis. Extremal probabilistic problems and Hotelling's T^2 test under a symmetry condition. *Ann. Statist.*, 22(1):357–368, 1994.

[8] I. Pinelis. Optimum bounds for the distributions of martingales in Banach spaces. *Ann. Probab.*, 22(4):1679–1706, 1994.

[9] I. Pinelis. Optimum bounds on moments of sums of independent random vectors. *Siberian Adv. Math.*, 5(3):141–150, 1995.

[10] I. Pinelis. Optimal tail comparison based on comparison of moments. In *High dimensional probability (Oberwolfach, 1996)*, volume 43 of *Progr. Probab.*, pages 297–314. Birkhäuser, Basel, 1998.

[11] I. Pinelis. Spherically symmetric functions with a convex second derivative and applications to extremal probabilistic problems. *Math. Inequal. Appl.*, 5(1):7–26, 2002.

[12] I. Pinelis. Exact inequalities for sums of asymmetric random variables, with applications. *Probab. Theory Related Fields*, 139(3-4):605–635, 2007.

[13] I. Pinelis. Optimal two-value zero-mean disintegration of zero-mean random variables. *Electron. J. Probab.*, 14:no. 26, 663–727, 2009.

[14] I. F. Pinelis and S. A. Utev. Estimates of moments of sums of independent random variables. *Theory Probab. Appl.*, 29(3):574–577, 1984.

[15] I. F. Pinelis and S. A. Utev. Sharp exponential estimates for sums of independent random variables. *Theory Probab. Appl.*, 34(2):340–346, 1989.

[16] J. V. Prohorov. Extremal problems in limit theorems. In *Proc. Sixth All-Union Conf. Theory Prob. and Math. Statist. (Vilnius, 19 60) (Russian)*, pages 77–84. Gosudarstv. Izdat. Političesk. i Naučn. Lit. Litovsk. SSR, Vilnius, 1962.

[17] H. P. Rosenthal. On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math.*, 8:273–303, 1970.

[18] S. A. Utev. Extremal problems in moment inequalities. In *Limit theorems of probability theory*, volume 5 of *Trudy Inst. Mat.*, pages 56–75, 175. "Nauka" Sibirsk. Otdel., Novosibirsk, 1985.